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ON SKOLEM AND HERBRAND THEOREMS FOR INTUITIONISTIC LOGIC

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I. INTRODUCTION

I have previously given Skolem and Herbrand theorems for various classical logics [3,4,5]. In this paper I will consider intuitionistic logic. Mints has given a counterexample to the ordinary Herbrand theorem [6]. His example is $\forall x \forall y(Fax \vee Fby) \rightarrow \exists z \exists u\, Fzu$. The formula is easily seen to be not provable in LI but both its Skolem transform and its Herbrand transforms are.

In this paper I hope to clear up the matter further. As in LK I will split the theory up into two parts - a Skolem theory and an Herbrand theory. The difficulties will be seen to only come in in the Skolem theory. It is not true in general that the Skolem morphisms are falsifiability morphisms. Everything else works as in LK .

II. THE FORMAL SYSTEM LI .

We give a sequential formulation of intuitionistic logic LI, following the development of LK [3]:

LANGUAGE

Propositional constant $\curlywedge$ (the falsity)

Connectives $\mathbb{M}$, $\mathbb{W}$, $\supset$

Quantifiers $\forall$, $\exists$

Parameters $a_1, a_2, ---, b, c, ---$

Variables $x_1, x_2, ---, y, z, ---$

Functionsymbols and constants e, $f_1, f_2, ---, g, h, ---$

Predicate symbols $P_1, P_2, ---, Q, R, ---$

In the usual way we build up

Terms $t_1, t_2, ---, u, v, ---$

Atomic formulae $A_1, A_2, ---, B, C, ---$

Finite (and empty) sequences of formulae $\Gamma_1, \Gamma_2, --, \Delta, \Lambda, --$

Sequents $\Gamma \rightarrow \Delta$

THE CALCULUS LI . On the language we build the sequential calculus LI in the usual way:

AXIOMS: $\Gamma_1, A, \Gamma_2 \rightarrow \Delta_1, A, \Delta_2$ for A atomic

$\Gamma_1, \curlywedge, \Gamma_2 \rightarrow \Delta$

STRUCTURAL RULES:

Permutation $\dfrac{\Gamma \rightarrow \Delta}{\Gamma^* \rightarrow \Delta^*}$, where Γ^* is obtained from Γ by a permutation of formulae, and similarly Δ^* from Δ .

Thinning $\dfrac{\Gamma \to \Delta}{\Gamma, \Gamma_1 \to \Delta, \Delta_1}$

Contraction $\dfrac{\Gamma, F, F \to \Delta}{\Gamma, F \to \Delta}$ $\dfrac{\Gamma \to G, G, \Delta}{\Gamma \to G, \Delta}$

Trivial rule $\dfrac{\Gamma \to \Delta \quad ---- \quad \Gamma \to \Delta}{\Gamma \to \Delta}$, where we have finite branching

LOGICAL RULES:

$\bigwedge \to$ $\dfrac{\Gamma, F_j \to \Delta}{\Gamma, \bigwedge_{i \in I} F_i \to \Delta}$ where F_j is one of the conjuncts in $\bigwedge_{i \in I} F_i$

$\to \bigwedge$ $\dfrac{--- \ \Gamma \to F_j, \Delta \ ---}{\Gamma \to \bigwedge_{i \in I} F_i, \Delta}$ where we have as premisses $\Gamma \to F_j, \Delta$ for all $j \in I$.

$\bigvee \to$ $\dfrac{--- \ \Gamma, F_j \to \Delta \ ---}{\Gamma, \bigvee_{i \in I} F_i \to \Delta}$ as $\to \bigwedge$

$\to \bigvee$ $\dfrac{\Gamma \to F_j, \Delta}{\Gamma \to \bigvee_{i \in I} F_i, \Delta}$ as $\bigwedge \to$

$\supset \to$ $\dfrac{\Gamma \to F, \Delta \qquad \Gamma, G \to \Delta}{\Gamma, F \supset G \to \Delta}$

$\to \supset$ $\dfrac{\Gamma, F \to G}{\Gamma \to F \supset G}$

$\forall \to$ $\dfrac{\Gamma, Ft \to G}{\Gamma, \forall x F x \to G}$ t is a term

$\to \forall$ $\dfrac{\Gamma \to Fa}{\Gamma \to \forall x F x}$ a is a parameter not in $\Gamma \to \forall x F x$

$\exists \rightarrow$	$\dfrac{\Gamma, Fa \rightarrow \Delta}{\Gamma, \exists x Fx \rightarrow \Delta}$	a is a parameter not in $\Gamma, \exists x Fx \rightarrow \Delta$
$\rightarrow \exists$	$\dfrac{\Gamma \rightarrow Ft, \Delta}{\Gamma \rightarrow \exists x Fx, \Delta}$	t is a term

This completes the description of our formal system LI . Note the succedents in $\rightarrow \supset$ and $\rightarrow \forall$.

III. COMPLETENESS THEOREM

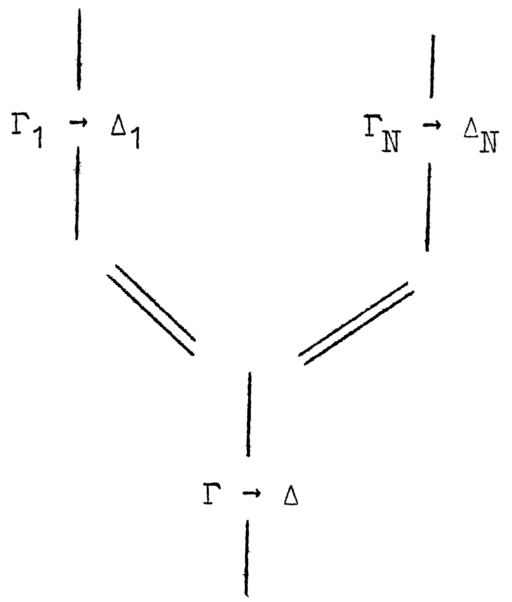
There are a number of completenesstheorems in the littera- ture. Our approach below is similar to the ones of Fitting [1], Görnemann [2] and Schütte [7].

We want to construct trees corresponding to classical trees. In LK it did not make any difference which formula we analyzed first as long as we did the analysis in a systematic way. In LI it makes a lot of difference. The reason is that in analyzing $\supset$ and $\forall$ in succedent we lose information. To take care of that we construct not only a tree but a tree of trees (or a forest of trees as we shall say).

DEFINITION Fix a sequent. A forest of trees over the sequent will be a tree of trees of sequents with the one sequent tree of the given sequent at the bottom node. Between each tree in the forest and its successortrees we have the following 4 possibili- ties:

(We write single lines as indicating the branches in the trees and double lines for the branches in the forest.)

1.



Here we have the same trees above except for $\Gamma_1 \to \Delta_1, \ldots, \Gamma_N \to \Delta_N$ instead of $\Gamma \to \Delta$. $\Gamma_1 \to \Delta_1, \ldots, \Gamma_N \to \Delta_N$ and $\Gamma \to \Delta$ are connected as one of the rules of LI except the rules $\to \supset$, $\to \forall$.

2.

$$\Gamma, F \to G$$

$$\Gamma \to \Delta$$

$$\Gamma \to F \supset G, \Delta$$

The tree above is the same as the one below except that we have $\Gamma \to \Delta$ instead of $\Gamma \to F \supset G, \Delta$ and we have added a new successor node to $\Gamma \to \Delta$ with sequent $\Gamma, F \to G$.

3.

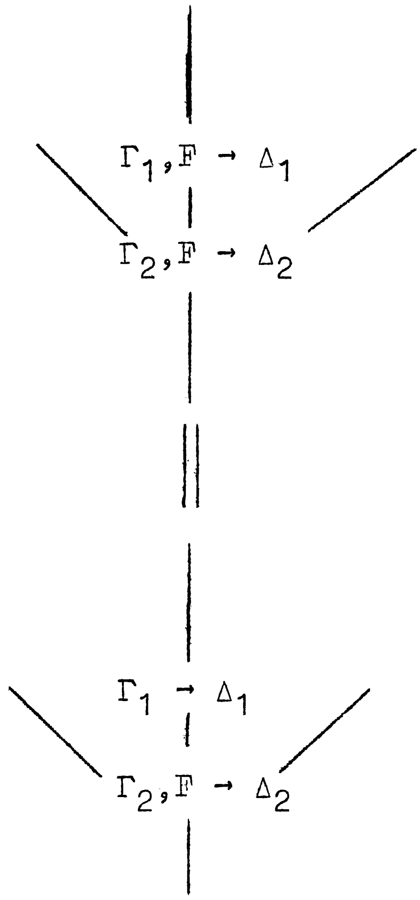
$$\Gamma \to Fa$$

$$\Gamma \to \Delta$$

$$\Gamma \to \forall x F x, \Delta$$

The tree above is the same as the one below except that we have $\Gamma \to \Delta$ instead of $\Gamma \to \forall x F x, \Delta$ and we have added a new successor node to $\Gamma \to \Delta$ with sequent $\Gamma \to Fa$, where a is a parameter not in $\Gamma \to \forall x F x$.

4.



Here $\Gamma_1 \to \Delta_1$ in the tree below is at one of the successornodes to $\Gamma_2, F \to \Delta_2$.

The tree above is the same as the one below except that we have $\Gamma_1, F \to \Delta_1$ instead of $\Gamma_1 \to \Delta_1$.

This concludes the definition of forest of trees.
It is with possibilities 2 and 3 that we build new nodes. Possibility 4 is an auxiliary construction. We will show below that we do not need possibility 4 in finite secured intuitionistic forests. More about this later.

For the forests we develop some of the usual theory.

PRECEDES, SUCCEEDS. As in LK [3], we give 'immediately precedes as formula / formulapart' and can hence define 'precedes as formula', 'precedes as formulapart', 'succeeds as formula', 'succeeds as formulapart', 'in the same strand', 'analysis'.

POSITIVE AND NEGATIVE We define positive and negative occurrences in a sequent $\Gamma \to \Delta$ inductively by:

i) Any formula in Δ occurs positively in $\Gamma \to \Delta$.

ii) Any formula in Γ occurs negatively in $\Gamma \to \Delta$.

iii) If $\bigwedge F_i$ or $\bigvee F_i$ occurs positively (negatively) in $\Gamma \to \Delta$, then each F_i occurs positively (negatively) in $\Gamma \to \Delta$.

iv) If $F \supset G$ occurs positively (negatively) in $\Gamma \to \Delta$, then F occurs negatively (positively) and G occurs positively (negatively) in $\Gamma \to \Delta$.

v) If $\forall xFx$ or $\exists xFx$ occurs positively (negatively) in $\Gamma \to \Delta$, then Fx occurs positively (negatively) in $\Gamma \to \Delta$.

GENERAL AND RESTRICTED $\forall x$ in $\Gamma \to \Delta$ is general (restricted) if it occurs positively (negatively) as $\forall xFx$ in $\Gamma \to \Delta$. $\exists xFx$ is restricted (general) if it occurs positively (negatively) as xFx in $\Gamma \to \Delta$.

NODES We assume that we have a system of notation for nodes in trees. We write $\leq$ for the usual tree-ordering of nodes.

TERMS BELONGING TO A NODE Given a forest $\mathcal{F}$ of trees over $\Gamma \to \Delta$. A term t belongs to the node ν in $\mathcal{F}$ if:

i) The node ν occurs in one of the trees of $\mathcal{F}$.

ii) t is built up from

a) symbols from $\Gamma \to \Delta$

b) the constant e

c) parameters introduced by $\to \forall$ or $\exists \to$ at nodes ν

INTUITIONISTIC FOREST An intuitionistic forest $\mathcal{F}$ over a sequent $\Gamma \to \Delta$ is a forest of trees over $\Gamma \to \Delta$ such that

i) the terms introduced by $\forall \to$ or $\to \exists$ at a node in $\mathcal{F}$ belong to the node in $\mathcal{F}$;

ii) parameters introduced in $\to \forall$ or $\exists \to$ are distinct if we analyze quantifiers not in the same strand or with distinct analysis or in distinct nodes, and

iii) there is a well-order of the parameters such that for any parameter a introduced by $\to \forall$ or $\exists \to$ in tree T in $\mathcal{F}$ all parameters occurring in trees below T are less than a (in the well-order).

Conditions ii (with a change) and iii are well known from our previous papers [3,4,5]. Condition i is new. Observe that the further up a node is, the more terms belong to it. In the countermodels to a sequent $\Gamma \to \Delta$, with $\nvdash_{LI} \Gamma \to \Delta$, we produce below, the terms belonging to a node will constitute the domain at the node. To prove "There is a secured intuitionistic forest over $\Gamma \to \Delta \Longrightarrow\ \vdash_{LI} \Gamma \to \Delta$ " it is necessary that the terms at a node ν does not depend on parameters introduced by $\to \forall$ or $\exists \to$ in nodes above ν .

SECURED A node in a tree is secured if it contains an axiom. A tree is secured if it contains a secured node. A branch in a forest is secured if it contains a secured tree. A forest is secured if all its branches are.

FINITENESS LEMMA For any sequent $\Gamma \to \Delta$, if there is a secured intuitionsistic forest over $\Gamma \to \Delta$,then there is a finite secured intuitionistic forest over $\Gamma \to \Delta$.

STANDARD FOREST   A forest is standard if the nodes are only connected by possibilities 1,2,3. i.e. we do not need possibility 4.

PROVABILITY LEMMA   If we have a finite secured standard forest over $\Gamma \to \Delta$, then $\vdash_{LI} \Gamma \to \Delta$.

Proof:

We can assume that all the top-most trees in the forest $\mathcal{F}$ are secured.

By induction starting with the top-most trees we then prove that every tree contains a sequent provable in LI .

So for example say we have used possibility 1 and that all the trees above contain LI-provable sequents. We must prove that the tree below contains an LI-provable sequent.

Now either we have an LI-provable sequent in one of the trees above which is also in the tree below, or $\vdash_{LI} \Gamma_1 \to \Delta_1, ---, \vdash_{LI} \Gamma_N \to \Delta_N$. But in the last case $\vdash_{LI} \Gamma \to \Delta$.

QED

The task now is to prove that if we have a finite secured intuitionistic forest over $\Gamma \to \Delta$, then there is a finite sequced standard intuitionistic forest over $\Gamma \to \Delta$.

The trick is to show that we can to some extent permute the applications of the possibilities 1,2,3,4. We formulate the lemmata on permutations of the applications a little sloppy but will show with examples below how to make it precise.

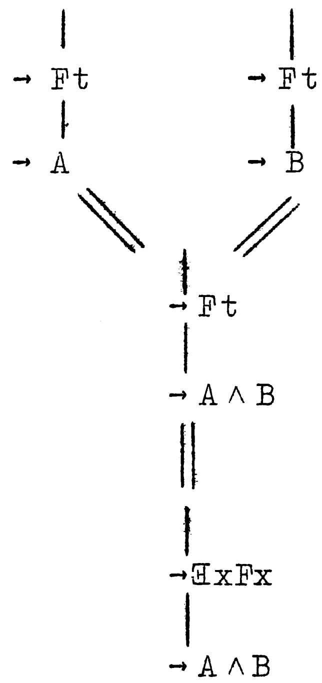
LEMMA 1   If we have an application of possibility 1 at node ν immediately after any other application at node μ when not

$\mu \preccurlyeq \nu$ then we can permute the applications.

LEMMA 2 If we have an application of possibility 1 at node ν immediately after an application of 2,3,4 at node ν then we can permute the applications.

EXAMPLES

1.



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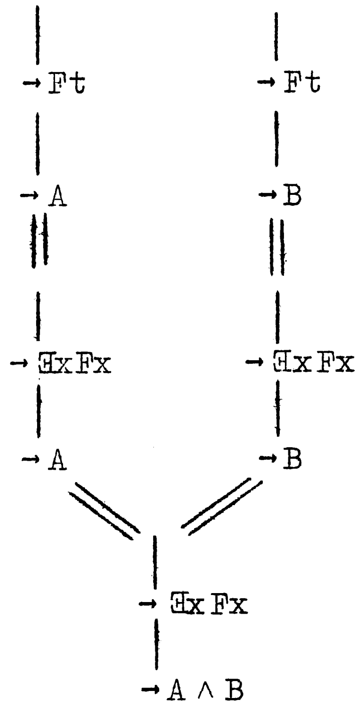

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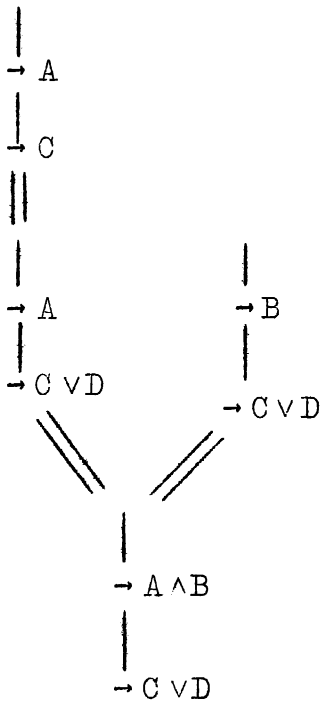
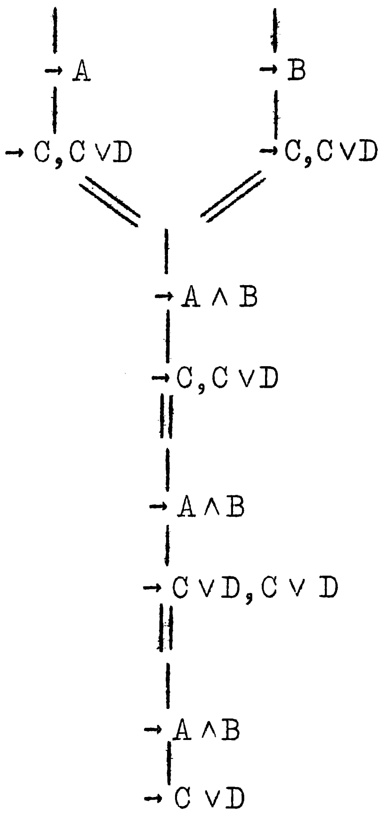


gives



3.

$$\begin{array}{ccc}
\rightarrow Fa & & \rightarrow Fa \\
| & & | \\
\rightarrow Gt & & \rightarrow Gt \\
\| & & \| \\
\rightarrow Fa & \text{gives} & \rightarrow \forall xFx \\
| & & | \\
\rightarrow \exists xGx & & \rightarrow Gt \\
\| & & \| \\
\rightarrow \forall xFx & & \rightarrow \forall xFx \\
| & & | \\
\rightarrow \exists xGx & & \rightarrow \exists xGx
\end{array}$$

Observe above that a cannot occur in t since a is introduced at a node which is above the node t belongs to. This is the place where the assumption on the terms introduced by $\rightarrow \exists$ or $\forall \rightarrow$ in intuitionistic forests comes in.
Using the permutations we get:

LEMMA Given a finite secured intuitionistic forest over $\Gamma \rightarrow \Delta$. There is then a finite secured intuitionistic forest $\mathcal{F}$ over $\Gamma \rightarrow \Delta$ such that any application of possibility 1 of node ν precedes all applications of possibility 7 at nodes $\prec \nu$, and precedes all applications of possibilities 2,3,4 at nodes $\preccurlyeq \nu$.

Observe now that if we have a forest as above then the applications of possibility 4 are of the form

$$\begin{array}{c}
\Gamma_2, F \rightarrow \Delta_2 \\
\Gamma_1, F \rightarrow \Delta_1 \\
\| \\
\Gamma_2 \rightarrow \Delta_2 \\
\Gamma_1, F \rightarrow \Delta_1
\end{array}$$

where F is already in Γ_2 or was in the antecedent of the node when it was formed by an application of possibility 2 or 3. That is we can change the application of possibility 4 wuth an application of possibility 1 (using contraction rule).

LEMMA Given a sequent. If we have a finite secured intuitionistic forest over $\Gamma \to \Delta$, then we can find a finite secured standard intuitionistic forest over $\Gamma \to \Delta$.

PROVABILITY THEOREM For any sequent $\Gamma \to \Delta$, if we have a secured intuitionistic forest over $\Gamma \to \Delta$, then $\vdash_{LI} \Gamma \to \Delta$.

With the provability theorem, the hard part is done towards the completeness theorem.

ANALYZING BRANCH A branch β in an intuitionistic forest $\mathcal{F}$ is an analyzing branch when:

i) if $\bigwedge F_i$ occurs in an antecedent at node ν in a tree of β, then each F_i occurs as a successor to $\bigwedge F_i$ in an antecedent at node ν in β;

ii) if $\bigwedge F_i$ occurs in a succeedent at node ν in β, then at least one F_i occurs as a sucessor to $\bigwedge F_i$ in a succeedent at node ν in β;

iii) if $W F_i$ occurs in an antecedent at node ν in β, then at least one F_i occurs as a successor to $W F_i$ in an antecedent at node ν in β;

iv) if $W F_i$ occurs in a succeedent at node ν in β, then each F_i occurs as a successor to $W F_i$ in a succeedent at node ν in β;

v) if $F \supset G$ occurs in an antecedent at node ν in β, then either F occurs as a successor to $F \supset G$ in a succeedent at node ν in β or G occurs as a successor to $F \supset G$ in an antecedent at node ν in β ;

vi) if $F \supset G$ occurs in a succedent at node ν in β , then there is a node $\mu \geqslant \nu$ such that F occurs in an antecedent and G in a succeedent as successors to $F \supset G$ at node μ in β ;

vii) if $\forall xFx$ occurs in an antecedent at node ν in β ; then for every term t belonging to ν in $\mathscr{F}$, Ft occurs in an antecedent at node ν in β ;

viii) if $\forall xFx$ occurs in a succeedent at node ν in β , then there is a node $\mu \geqslant \nu$ and a term t belonging to μ such that Ft occurs as a successor to $\forall xFx$ in a succeedent at node μ in β ;

ix) if $\exists xFx$ occurs in an antecedent at node ν in β ; then there is a term t belonging to ν such that Ft occurs as a successor to $\exists xFx$ in an antecedent at node ν in β ;

x) if $\exists xFx$ occurs in a succeedent at node ν in β , then for every term t belonging to ν , Ft occurs in a succeedent at node ν in β ; and

xi) if any formula F occurs in an antecedent at node ν in β and $\mu > \nu$, then there occurs an F as a successor in an antecedent at node μ in β .

ANALYZING FOREST An intuitionistic forest is analyzing if all its branches are.

With only the obvious changes from LK we prove:

ANALYZING LEMMA To any sequent we can find an analyzing intuitionistic forest over it.

FALSIFIABILITY LEMMA   If we have a not-secured analyzing branch β in an intuitionistic forest over $\Gamma \rightarrow \Delta$, then we can find a falsifying Kripke model of $\Gamma \rightarrow \Delta$.

Proof:

Assume β as above.

We construct the Kripke model as follows:

The modelstructure is given by the nodes occuring in β and the $\preccurlyeq$- ordering between them.

The domain at node ν is the set of terms belonging to node ν .

An atomic formula is true at node ν if and only if it occurs in an antecedent at ν in β .

By induction over the length of formulae we prove that every formula occuring at ν in β is true at ν if it occurs in an antecedent and false at ν if it occurs in a succeedent.

This gives the lemma.

QED

SOUNDNESS LEMMA   For any sequent $\Gamma \rightarrow \Delta$, if $\vdash_{LI} \Gamma \rightarrow \Delta$, then there are no falsifying Kripke-models of $\Gamma \rightarrow \Delta$.

COMPLETENESS THEOREM   For any sequent $\Gamma \rightarrow \Delta$, $\vdash_{LI} \Gamma \rightarrow \Delta$ if and only if there are no falsifying Kripke-models of it.

CONSISTENCY THEOREM   For any sequent $\Gamma \rightarrow \Delta$ we have exactly one of i and ii below

i) a secured intuitionistic forest over $\Gamma \rightarrow \Delta$

ii) an intuitionistic forest over $\Gamma \rightarrow \Delta$ with not-secured analyzing branch.

IV. THE SKOLEM THEOREM

In this section we develop the Skolem theory. We can carry the Skolem morphisms and the proof that they are provability morphisms directly over from LK . But this is also all we can do. There are a number of obstructions to proving that they are falsifiability morphisms.

MORPHISMS An <u>intuitionistic morphism</u> is a transformation of intuitionistic forests into intuitionistic forests preserving the tree structure. A <u>provability morphism</u> is an intuitionistic morphism which transforms secured forests into secured forests. An <u>analyzing morphism</u> transforms analyzing forests into analyzing forests. A <u>falsifiability morphism</u> transforms analyzing not-secured forests into analyzing not-secured forests. An <u>intuitionistic isomorphism</u> is an intuitionistic morphism which is both a provability and a falsifiability morphism.

SKOLEM TRANSFORM As in LK [3] we define the Skolem transforms S_{π}^{x} and S .

SKOLEM MORPHISM As in LK [3] with the obvious changes we define $\mathcal{S}_{\pi}^{x}$ and $\mathcal{S}$.

Following [3] we prove

THEOREM $\mathcal{S}$ and $\mathcal{S}_{\pi}^{x}$ are provability morphisms.

We will now discuss the obstructions to proving that the Skolem-morphisms are intuitionistic isomorphisms.

There are two main obstructions:

i) $\mathfrak{S}$ is not in general an analyzing morphism.

ii) The strong analyzing lemma is not true in LI .

OBSTRUCTION 1 : Not analyzing morphism.

The problem comes in in our formulation of intuitionistic forest. At a node ν we are only allowed to put terms belonging to the node. Say we have a parameter a in an intuitionsistic forest $\mathcal{F}$ over $\Gamma \to \Delta$ not belonging to the bottom node. After performing $\mathfrak{S}$ a goes over into a term t built up from the constant e and symbols from $S(\Gamma \to \Delta)$. In $\mathfrak{S}(\mathcal{F})$ t belongs to the bottom mode and we should analyze restricted quantifiers in the bottom node with t . This is opposed to the sutuation in $\mathcal{F}$ where we should not analyze restricted quantifiers in the bottom node with a . A counterexample to the full Skolem theorem using this obstruction is $\forall x(A \vee Bx) \to A \vee \forall xBx$. The sequent is not provable in LI , but its Skolem transform $\forall x(A \vee Bx) \to$ $\to A \vee Bf$ is. If we write down an analyzing intuitionistic forest over $\forall x(A \vee Bx) \to A \vee \forall xBx$, it is easy to see what goes wrong.

Now we can prove that the sequent above is the only type of sequent that goes wrong under this obstruction. Namely:

i) Let LG be the sequential calculus we get by changing $\to \forall$ to

$$\to \forall' \quad \frac{\Gamma \to Fa, \Delta}{\Gamma \to \forall xFx, \Delta} \qquad \begin{array}{l} a \text{ does not occur} \\ \text{in } \Gamma \to \forall xFx, \Delta \end{array}$$

ii) LG is equivalent to LI + Cut +

+ the schema '$\forall x(A \vee Bx) \to A \vee \forall xBx$'.

iii) Görnemann has proved in her thesis that LG is complete for Kripke-models of constant domains [2]. This follows also from the below.

iv) Change the condition on the terms introduced by $\forall \rightarrow$ and $\rightarrow \exists$ in intuitionistic forest to be:
the terms shall be built up from symbols in $\Gamma \rightarrow \Delta$, the constant e , and parameters introduced by $\rightarrow \forall$ or $\exists \rightarrow$ somewhere in the forest.

Call the new forests LG-forests

v) In the work of Görnemann [2] she proves:
If we have a secured LG-forest over $\Gamma \rightarrow \Delta$,
then $\vdash_{LG} \Gamma \rightarrow \Delta$.

vi) With the obvious changes in the definition of analyzing branch and analyzing LG-forest we follow the theory above for LI .

vii) We get LG complete for Kripke-models of constant domain.

viii) Now following the theory of LK [3] we prove:
a. $\mathfrak{S}$ and $\mathfrak{S}^{x}_{\pi}$ are LG-provability morphisms.
b. $\mathfrak{S}$ and $\mathfrak{S}^{x}_{\pi}$ are LG-analyzing morphisms.

We are now left with

OBSTRUCTION 2 : The strong analyzing lemma is false.

This is a much more serious threat to our theory than obstruction 1. The only problem in the strong analyzing lemma comes in in the analysis of $\supset$ in the antecedent. It turns out that we are forced to analyze $\Gamma, F \supset G \to \Delta$ by $F \supset G, \Gamma, G \to \Delta$ and $\Gamma, F \supset G \to$ $\to F, A$. In the last sequents we see that we may hence have two copies of the same general quantifier in one occurrence of a node.

Instead of the strong analyzing lemma where we try to handle all general quantifiers sumultaneously we can make a lemma which handles one general quantifier at a time. Using such a lemma one can prove:

THEOREM $\Gamma \to \Delta$ is a sequent. Assume that the general quantifier x at position π in $\Gamma \to \Delta$, does not occur in a nagative subformula $F \supset G$ of $\Gamma \to \Delta$.
Then $\mathcal{S}^{x}_{\pi}$ is a falsifiability LG-morphism.
and $\mathcal{S}^{x}_{\pi}$ is an LG-isomorphism.

The proof is straightforward from the hints above. That the theorem is still not satifactory can be seen from the counterexamples to getting rid of the extra assumption:

$$\forall x \neg\neg Fx \to \neg\neg \forall x Fx$$

$$A \supset \exists x Fx \to \exists x (A \supset Fx)$$

Neither of the sequents are LG-provable but both have Skolem-transform which are. One could h e for the following approach to give a Skolem-theorem:

Construct a new logic by throwing in as axioms all sequents with LI-provable Skolem-transforms. As rules we could have the rules of LI. It is now a hope that we chould end up with a natural logic only changing the interpretations of $\forall$ and $\exists$ in LI . This hope is killed by the last sequent above, $A \supset \exists x Fx \to$ $\to \exists x (A \supset Fx)$. We would not want it provable since the corresponding propositional sequent

$$A \supset B \vee C \rightarrow (A \supset B) \vee (A \supset C)\,, \qquad \text{is not.}$$

Here we leave the Skolem-theory.

V. THE HERBRAND THEOREM

We now come to the Herbrand theorem for LI . Here we can follow the entire theory of LK with only the obvious changes [3] .

HERBRAND DOMAIN As in LK .

HERBRAND TRANSFORM $H^{x}_{\pi,\mathfrak{D}}$, $H_{\mathfrak{D}}$, H_n as in LK .

HERBRAND MORPHISM $\mathcal{H}^{x}_{\pi,\mathfrak{D}}$, $\mathcal{H}_{\mathfrak{D}}$, $\mathcal{H}_n$ as in LK with only the obvious changes.

As in LK the theorems are obvious.

THEOREM

$\mathcal{H}^{x}_{\pi,\mathfrak{D}}$, $\mathcal{H}_{\mathfrak{D}}$, $\mathcal{H}_n$ are falsifiability morphisms.

THEOREM If $\mathcal{D}_1 \subseteq \mathcal{D}_2$ and $\mathcal{H}^{x}_{\pi,\mathfrak{D}_1}(\mathcal{F})$ is a secure intuitionistic forest, then also $\mathcal{H}^{x}_{\pi,\mathfrak{D}_2}(\mathcal{F})$.

THEOREM If $\mathcal{F}$ is a secured intuitionistic forest, then there is $\mathcal{D}$ with $\mathcal{H}_{\mathcal{D}}(\mathcal{F})$ secured, and N with $\mathcal{H}_N(\mathcal{F})$ secured.

VI. CONCLUSION

As an interesting result we note that the Herbrand theory works as in classical logic. The problems come in the Skolem theory, and more specifically in $\not\vdash_{LI} \Gamma \to \Delta \Longrightarrow \not\vdash_{LI} S(\Gamma \to \Delta)$.

There are many ways to formulate what goes wrong. Here is one:

Consider the analyzing intuitionistic forests as the systematic way of producing possible falsifying Kripke-models. In such a falsifying Kripke-model we must give solutions to the general quantifiers. Now when we perform the Skolem morphism we forces a too strong uniformity on the solutions of the general quantifiers. (This is a way of expressing what goes wrong with obstruction 2 which is the most serious obstruction.)

In addition the Skolem morphisms forces us only to consider Kripke-models of constant domains as possible falsifying Kripke-models. (Here we have obstruction 1.)

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